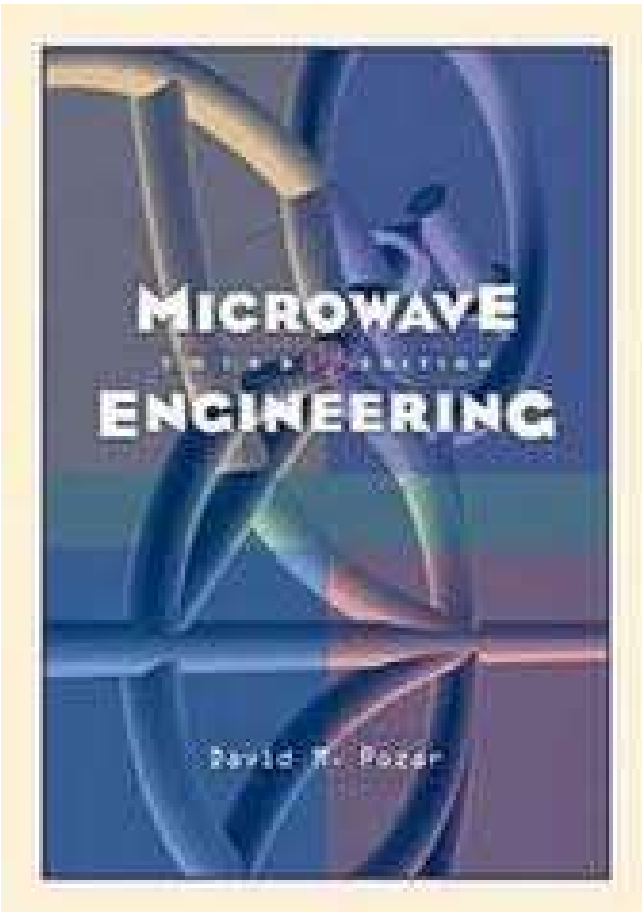
ECE 5317-6351
Microwave Engineering

Fall 2012

Prof. David R. Jackson
Dept. of ECE

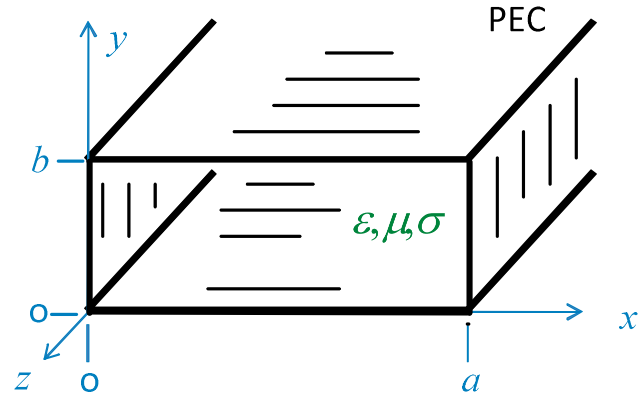


Notes 7

Waveguides Part 4: Rectangular and Circular Waveguide

Rectangular Waveguide

- One of the earliest waveguides.
- Still common for high power and high microwave / millimeter-wave applications.



- It is essentially an electromagnetic pipe with a rectangular cross-section.

Single conductor $\Rightarrow$ No TEM mode

For convenience

- $a \geq b$.
- the long dimension lies along x.

TE$_z$ Modes

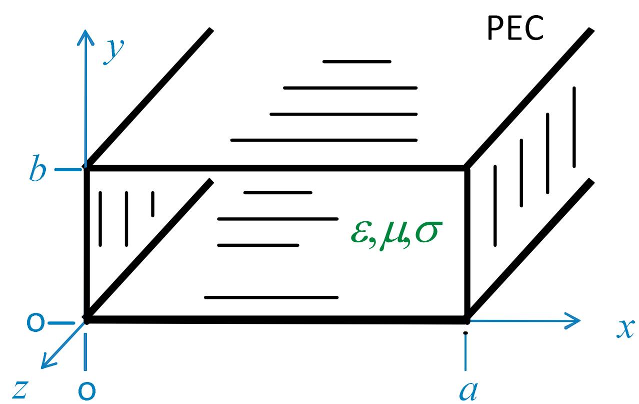
Recall

$$H_z(x,y,z) = h_z(x,y)e^{\mp jk_z z}$$

where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)h_z(x,y) = 0$$

$$k_c = \left(k^2 - k_z^2\right)^{1/2}$$



Subject to B.C.'s:

$$E_x = 0 \;\Rightarrow\; \frac{\partial H_z}{\partial y} \qquad @\, y = 0,\, b$$

and

$$E_y = 0 \;\Rightarrow\; \frac{\partial H_z}{\partial x} \qquad @\, x = 0,\, a$$

TE$_z$ Modes (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y)=-k_c^2\,h_z(x,y)$$ (eigenvalue problem)

Using *separation of variables*, let $h_z(x,y)=X(x)Y(y)$

$$\Rightarrow Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=-k_c^2XY$$

Must be a constant

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_c^2$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2$$ and $$\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2$$

where $k_x^2+k_y^2=k_c^2$ $\longleftarrow$ dispersion relationship

TE_z Modes (cont.)

Hence,

$$h_z(x,y) = \overbrace{(A\cos k_x x + B\sin k_x x)}^{X(x)}\overbrace{(C\cos k_y y + D\sin k_y y)}^{Y(y)}$$

Boundary Conditions:

$$\frac{\partial h_z}{\partial y} = 0 \qquad @ y = 0,\, b \qquad \text{A}$$

$$\frac{\partial h_z}{\partial x} = 0 \qquad @ x = 0,\, a \qquad \text{B}$$

A $\Rightarrow$ $D = 0$ and $k_y = \dfrac{n\pi}{b}$ $\quad n = 0,1,2,\ldots$

B $\Rightarrow$ $B = 0$ and $k_x = \dfrac{m\pi}{a}$ $\quad m = 0,1,2,\ldots$

$\Rightarrow$ $h_z(x,y) = A_{mn}\cos\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)$ and $k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2$

TE$_z$ Modes (cont.)

Therefore,

$$H_z = A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$k_z = \sqrt{k^2 - k_c^2}$$
$$= \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

From the previous field-representation equations, we can show

$$E_x = \frac{j\omega\mu n\pi}{k_c^2 b}A_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$E_y = -\frac{j\omega\mu m\pi}{k_c^2 a}A_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$H_x = \pm\frac{jk_z m\pi}{k_c^2 a}A_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$H_y = \pm\frac{jk_z n\pi}{k_c^2 b}A_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

Note:

$$m = 0,1,2,\ldots$$
$$n = 0,1,2,\ldots$$

But $m = n = 0$ is not allowed!

(non-physical solution)

$$\Rightarrow \underline{H} = \underline{\hat{z}}\,A_{00}e^{\mp jkz};\ \nabla\cdot\underline{H} \neq 0$$

TE$_z$ Modes (cont.)

Lossless Case $(\varepsilon_c = \varepsilon = \varepsilon')$

$$k_z^{mn} = \sqrt{k^2 - \left(k_c^{mn}\right)^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$\Rightarrow \mathrm{TE}_{mn}$ mode is at cutoff when $k = k_c^{mn}$

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Lowest cutoff frequency is for TE$_{10}$ mode $(a > b)$

We will revisit this mode.

$$f_c^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Dominant TE mode (lowest f_c)

TE_z Modes (cont.)

At the cutoff frequency of the TE_{10} mode (lossless waveguide):

$$f_c^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

$$\Rightarrow \quad \lambda = \frac{c_d}{f} = \frac{c_d}{f_c^{10}} = \frac{c_d}{\dfrac{1}{2a\sqrt{\mu\varepsilon}}} = 2a$$

$$\boxed{a\Big|_{f=f_c} = \lambda_d / 2}$$

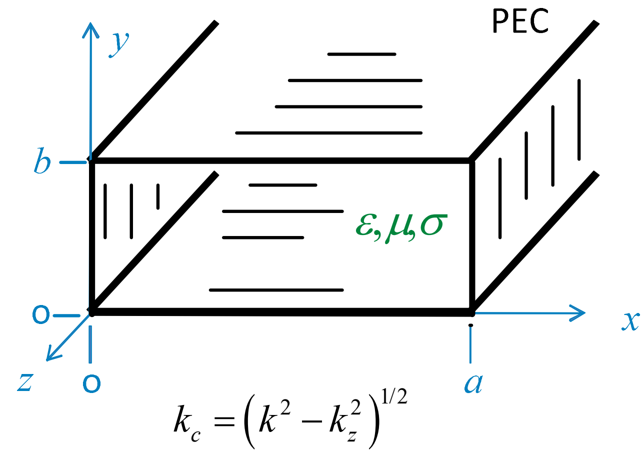
For a given operating wavelength (corresponding to $f > f_c$) , the dimension a must be at least this big in order for the TE_{10} mode to propagate.

Example: Air-filled waveguide, $f = 10$ GHz. We have that $a > 3.0$ cm/2 = 1.5 cm.

TM_z Modes

Recall

$$E_z(x,y,z) = e_z(x,y)e^{\mp jk_z z}$$



where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)e_z(x,y) = -k_c^2 e_z(x,y)$$ (eigenvalue problem)

Subject to B.C.'s: $E_z = 0$ @ $x = 0,\ a$

@ $y = 0,\ b$

Thus, following same procedure as before, we have the following result:

TM$_z$ Modes (cont.)

$$e_z(x,y) = \overbrace{(A\cos k_x x + B\sin k_x x)}^{X(x)}\overbrace{(C\cos k_y y + D\sin k_y y)}^{Y(y)}$$

Boundary Conditions: $e_z = 0$ $@\, y = 0,\, b$ A

$@\, x = 0,\, a$ B

A $\Rightarrow$ $C = 0$ and $k_y = \dfrac{n\pi}{b}$ $n = 0, 1, 2, \ldots$

B $\Rightarrow$ $A = 0$ and $k_x = \dfrac{m\pi}{a}$ $m = 0, 1, 2, \ldots$

$\Rightarrow$ $e_z = B_{mn}\sin\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right)$ and $k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2$

TM$_z$ Modes (cont.)

Therefore

$$E_z = B_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$k_z = \sqrt{k^2 - k_c^2}$$
$$= \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

From the previous field-representation equations, we can show

$$m=1,2,3,\ldots$$
$$n=1,2,3,\ldots$$

$$H_x = \frac{j\omega\varepsilon_c n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$H_y = -\frac{j\omega\varepsilon_c m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$E_x = \mp\frac{jk_z m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$E_y = \pm\frac{jk_z n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

Note: If either m or n is zero, the field becomes a trivial one in the TM$_z$ case.

TM_z Modes (cont.)

Lossless Case $(\varepsilon_c = \varepsilon = \varepsilon')$

$$\beta_{mn} = \sqrt{k^2 - \left(k_c^{mn}\right)^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

(same as for TE modes)

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

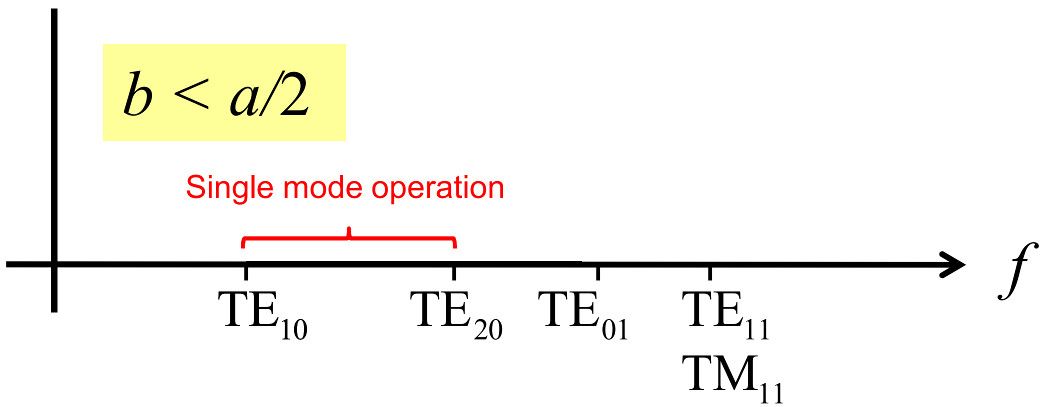
Lowest cutoff frequency is for the TM_{11} mode

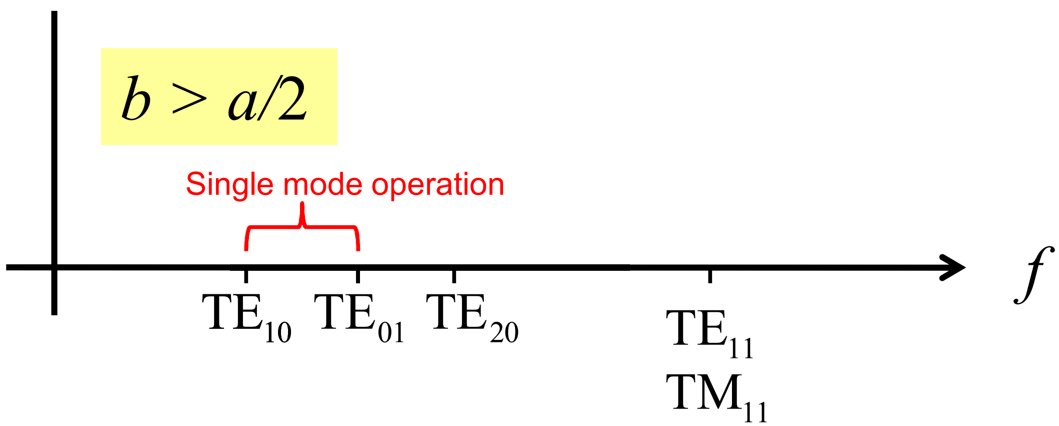
$$f_c^{11} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

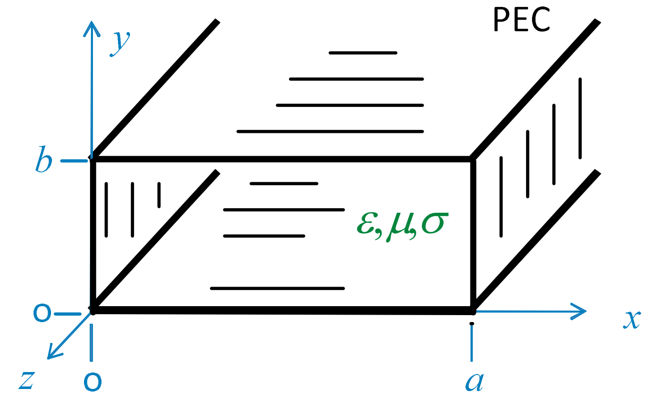
Dominant TM mode (lowest f_c)

Mode Chart

Two cases are considered:







$a > b$

The maximum band for single mode operation is $2 f_c^{10}$.

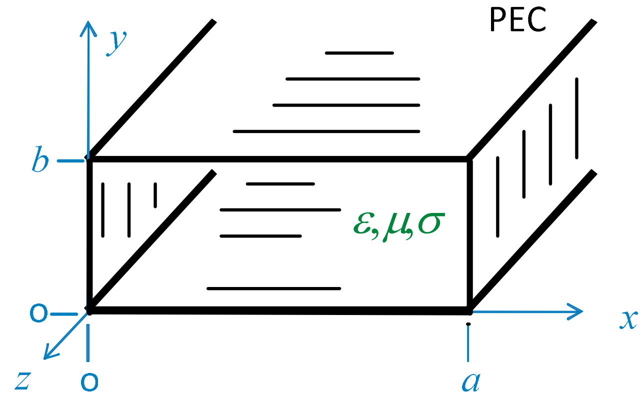
$$\Rightarrow b \le a/2$$

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Dominant Mode: TE_{10} Mode

For this mode we have

$$m = 1,\ \ n = 0,\ \ k_c = \frac{\pi}{a}$$



Hence we have

$$H_z = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{\mp j k_z z}$$

$$H_x = \pm j \frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{\mp j k_z z}$$

$$E_y = \underbrace{-\frac{j\omega\mu a}{\pi} A_{10}}_{E_{10}} \sin\left(\frac{\pi}{a} x\right) e^{\mp j k_z z}$$

$$E_x = E_z = H_y = 0$$

$$k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$E_y = E_{10} \sin\left(\frac{\pi}{a} x\right) e^{\mp j k_z z}$$

$$A_{10} = \frac{-\pi}{j\omega\mu a} E_{10}$$

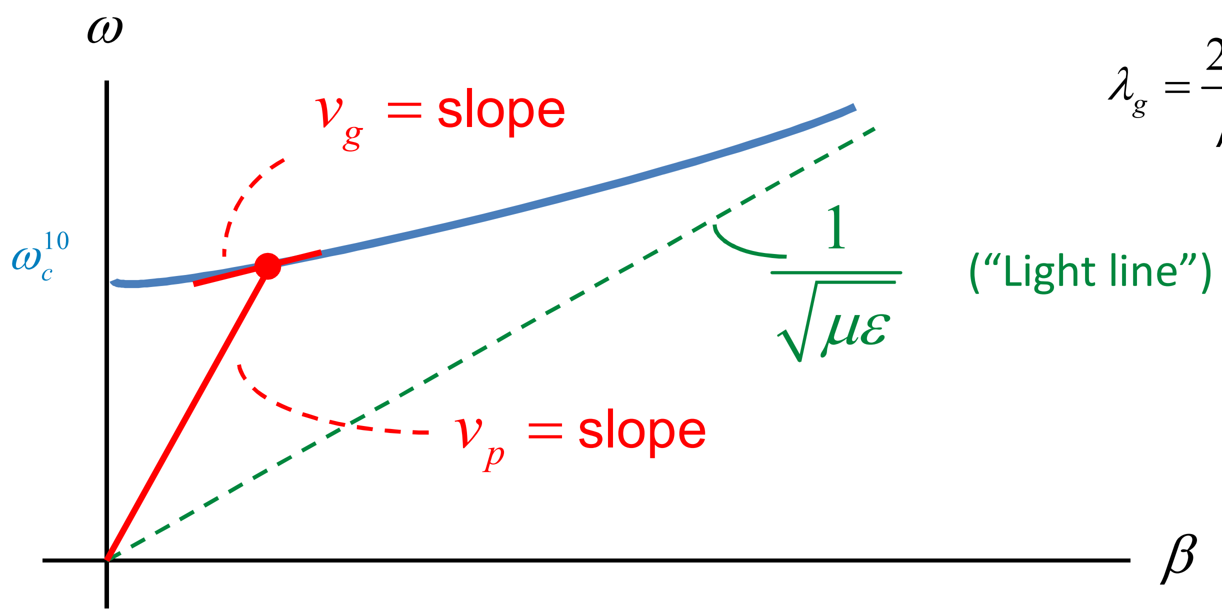
Dispersion Diagram for TE_{10} Mode

Lossless Case $(\varepsilon_c = \varepsilon = \varepsilon')$

$f > f_c$

$$k_z = \beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$\lambda_g = \frac{2\pi}{\beta}$$

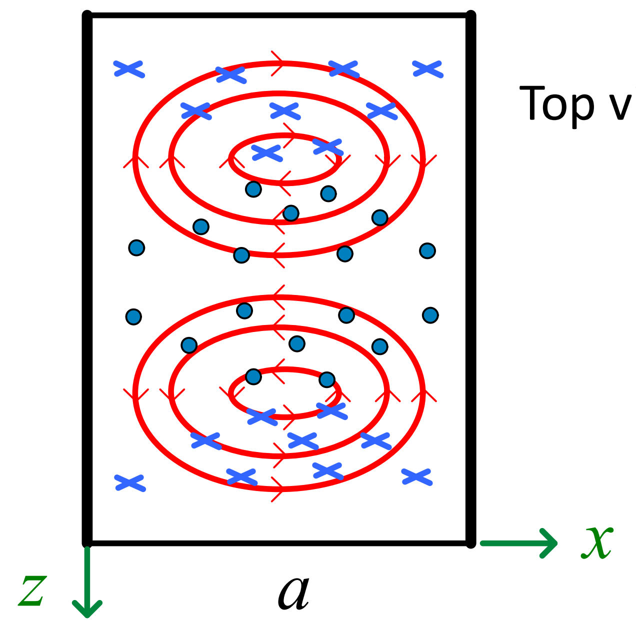


Phase velocity: $v_p = \dfrac{\omega}{\beta}$

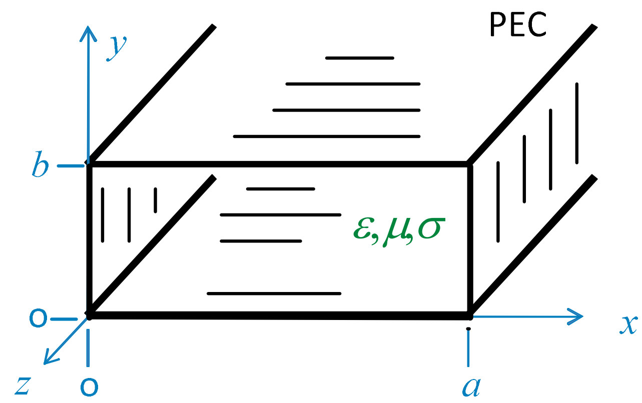
Group velocity: $v_g = \dfrac{d\omega}{d\beta}$

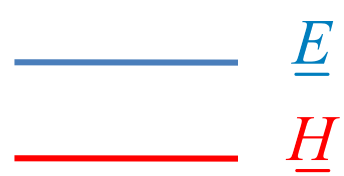
Velocities are slopes on the dispersion plot.

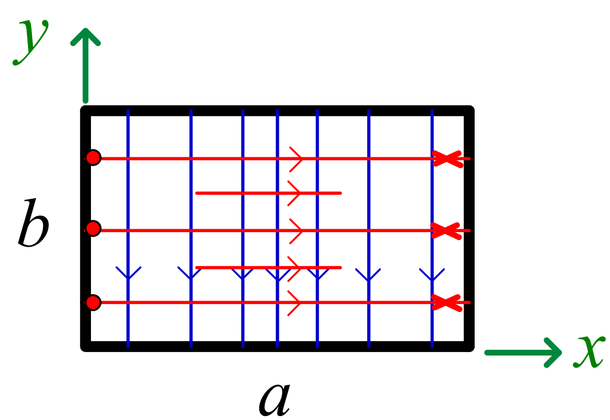
Field Plots for TE$_{10}$ Mode



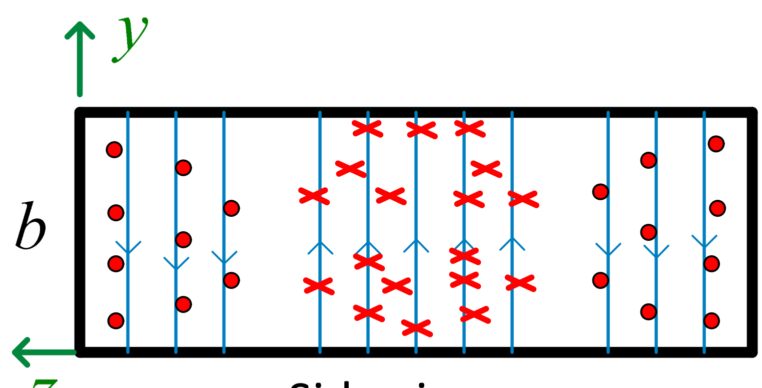
Top view





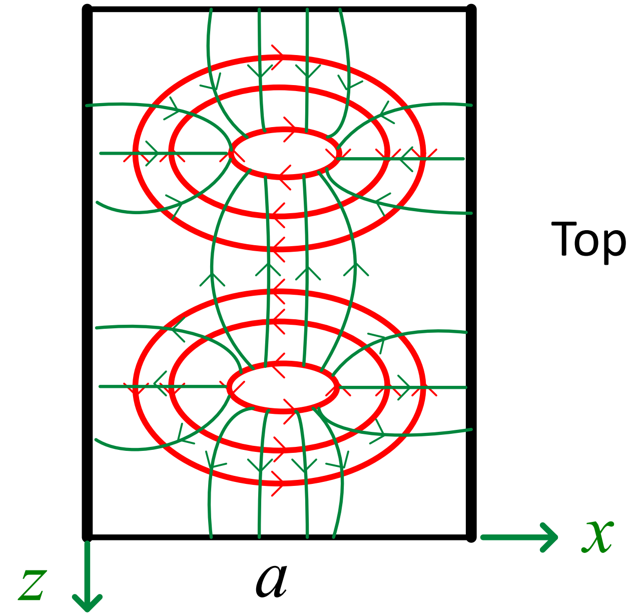


End view

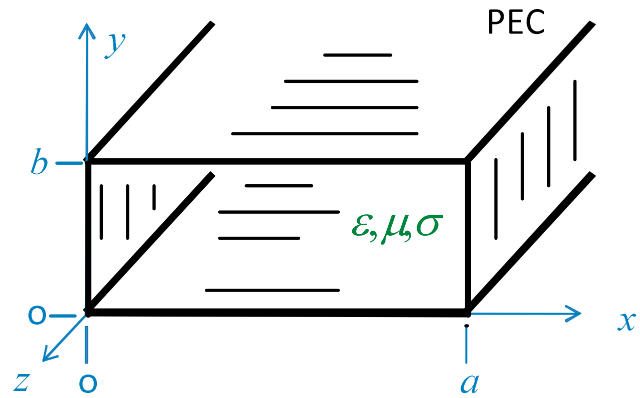


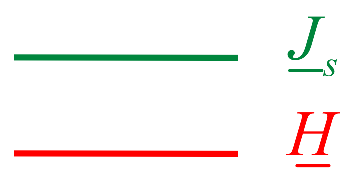
Side view

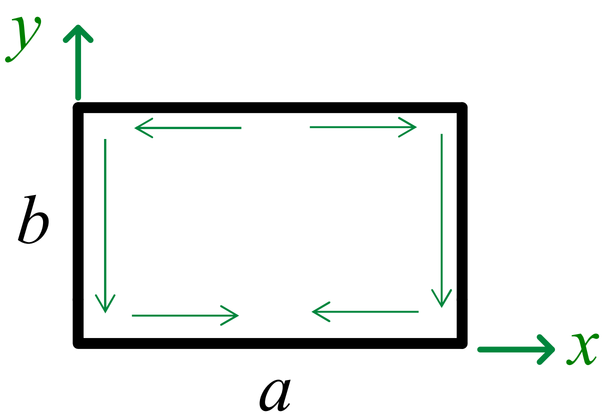
Field Plots for TE$_{10}$ Mode (cont.)



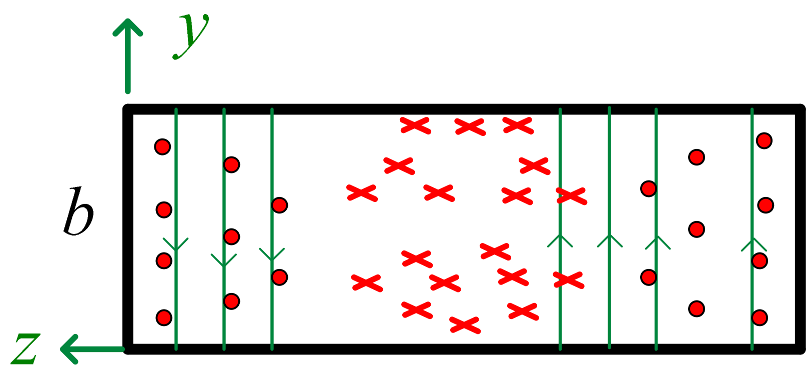
Top view







End view



Side view

Power Flow for TE_{10} Mode

Time-average power flow in the z direction:

$$P_{10}^{+} = \frac{1}{2}\mathrm{Re}\left\{\int_0^a\int_0^b \left(\underline{E}\times\underline{H}^*\right)\cdot\underline{\hat{z}}\,dydx\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\int_0^a\int_0^b -E_y H_x^*\,dydx\right\}$$

$$= \frac{\omega\mu a^3\left|A_{10}\right|^2 b}{4\pi^2}\mathrm{Re}\left\{k_z\right\}$$

Note:

$$\int_0^a\int_0^b \sin^2\left(\frac{\pi x}{a}\right)dydx = \frac{ab}{2}$$

In terms of amplitude of the field amplitude, we have

$$P_{10}^{+} = \left(\frac{ab}{4\omega\mu}\right)\mathrm{Re}\left\{k_z\right\}\left|E_{10}\right|^2 \qquad A_{10} = \frac{-\pi}{j\omega\mu a}E_{10}$$

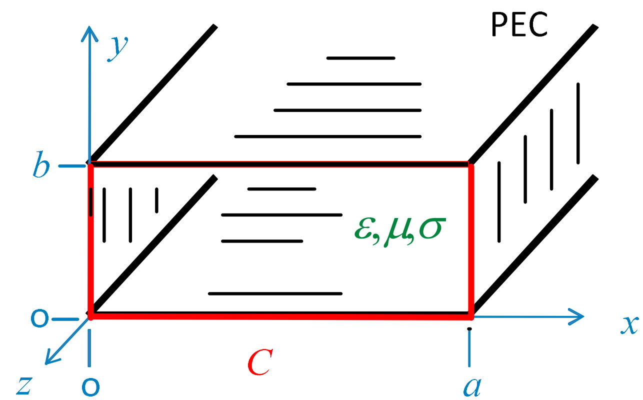
For a given maximum electric field level (e.g., the breakdown field), the power is increased by increasing the cross-sectional area (ab).

Attenuation for TE$_{10}$ Mode

Recall $\alpha_c = \dfrac{P_l(0)}{2P_0}$ $\quad P_0 = P_{10}^{+}$ (calculated on previous slide)

$$P_l(0) = \frac{R_s}{2} \int_C \left| \underline{J}_s \right|^2 d\ell$$

$\underline{J}_s = \underline{\hat{n}} \times \underline{H}$ on conductor



$$H_z^{+} = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$H_x^{+} = j\frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

Side walls

$$@ \;\; x = 0: \;\; \underline{J}_s^{side} = \underline{\hat{x}} \times \underline{H}\Big|_{x=0} = -\underline{\hat{y}} H_z = -\underline{\hat{y}} A_{10}\, e^{-jk_z z}$$

$$@ \;\; x = a: \;\; \underline{J}_s^{side} = -\underline{\hat{x}} \times \underline{H}\Big|_{x=a} = \underline{\hat{y}} H_z = -\underline{\hat{y}} A_{10}\, e^{-jk_z z}$$

$$\Rightarrow J_{sy}^{side} = -A_{10}\, e^{-jk_z z}$$

Attenuation for TE$_{10}$ Mode (cont.)

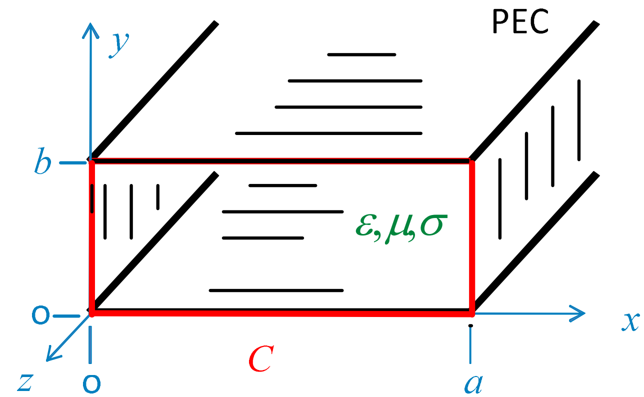
Top and bottom walls

$$@\ \ y=0:\ \ \underline{J}_s^{bot} = \underline{\hat{y}} \times \underline{H}\Big|_{y=0}$$

$$@\ \ y=b:\ \ \underline{J}_s^{top} = -\underline{\hat{y}} \times \underline{H}\Big|_{y=b}$$

$$\underline{J}_s^{top} = -\underline{J}_s^{bot}$$

(since fields of this mode are independent of y)



$$H_z^+ = A_{10}\cos\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

$$H_x^+ = j\frac{k_z a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

$$\Rightarrow P_l(0) = 2\left(\frac{R_s}{2}\int_0^b \left|\underline{J}_s^{side}\right|^2 dy + \frac{R_s}{2}\int_0^a \left|\underline{J}_s^{top}\right|^2 dx\right)$$

$$= R_s\int_0^b \left|J_{sy}^{side}\right|^2 dy + R_s\int_0^a\left(\left|J_{sx}^{top}\right|^2 + \left|J_{sz}^{top}\right|^2\right)dx$$

$$= R_s\left|A_{10}\right|^2\left(b + \left(\frac{\left|k_z^2\right|a^3}{2\pi^2} + \frac{a}{2}\right)\right)$$

$$J_{sz}^{top} = H_x \qquad J_{sx}^{top} = -H_z$$

$$\Rightarrow J_{sz}^{top} = j\frac{k_z a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

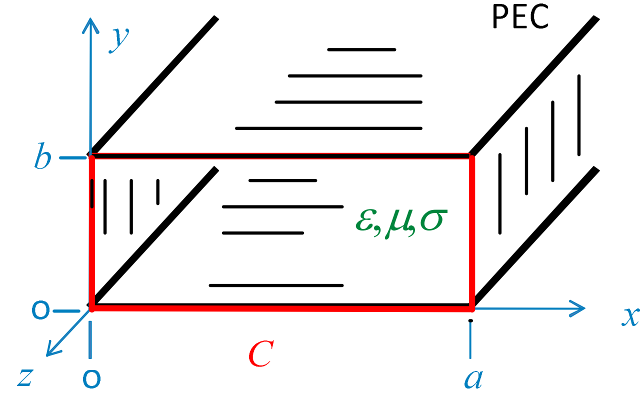
$$J_{sx}^{top} = A_{10}\cos\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

Attenuation for TE$_{10}$ Mode (cont.)

Assume $f > f_c$

$$k_z \approx \beta$$

(The wavenumber is taken as that of a guide with perfect walls.)



$$P_l(0) = R_s \left|A_{10}\right|^2 \left(b + \left(\frac{\beta^2 a^3}{2\pi^2} + \frac{a}{2} \right) \right)$$

$$P_{10}^{+} = \left(\frac{ab}{4\omega\mu} \right) \beta \left|E_{10}\right|^2$$

$$E_{10} = -\frac{j\omega\mu a A_{10}}{\pi}$$

Simplify, using $\beta^2 = k^2 - k_c^2 \qquad k_c^{10} = \frac{\pi}{a}$

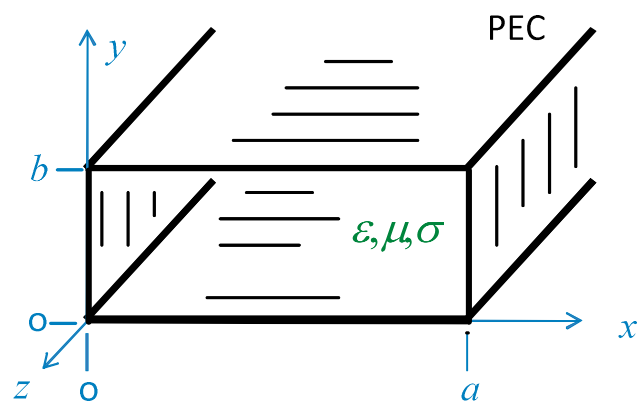
$$\alpha_c = \frac{P_l(0)}{2P_{10}^{+}}$$

Final result:

$$\alpha_c = \frac{R_s}{a^3 b \beta (k\eta)} \left(2b\pi^2 + a^3 k^2 \right) \quad [\text{np/m}]$$

Attenuation in dB/m

Let z = distance down the guide in meters.



$$\alpha_c\,[\mathrm{dB/m}] = -20\log_{10}\left(e^{-\alpha_c z}\right)/\,z$$

$$= \left(\alpha_c z\right)20\log_{10}(e)/\,z$$

$$= 8.686\alpha_c$$

Attenuation [dB/m]

Hence

$$\boxed{\mathrm{dB/m} = 8.686\ [\mathrm{np/m}]}$$

Attenuation for TE_{10} Mode (cont.)

Brass X-band air-filled waveguide

$$\left(\sigma \approx 2.6\times 10^{7}\ [\mathrm{S/m}]\right)$$

X band: $\approx 8-12\ [\mathrm{GHz}]$

(See the table on the next slide.)

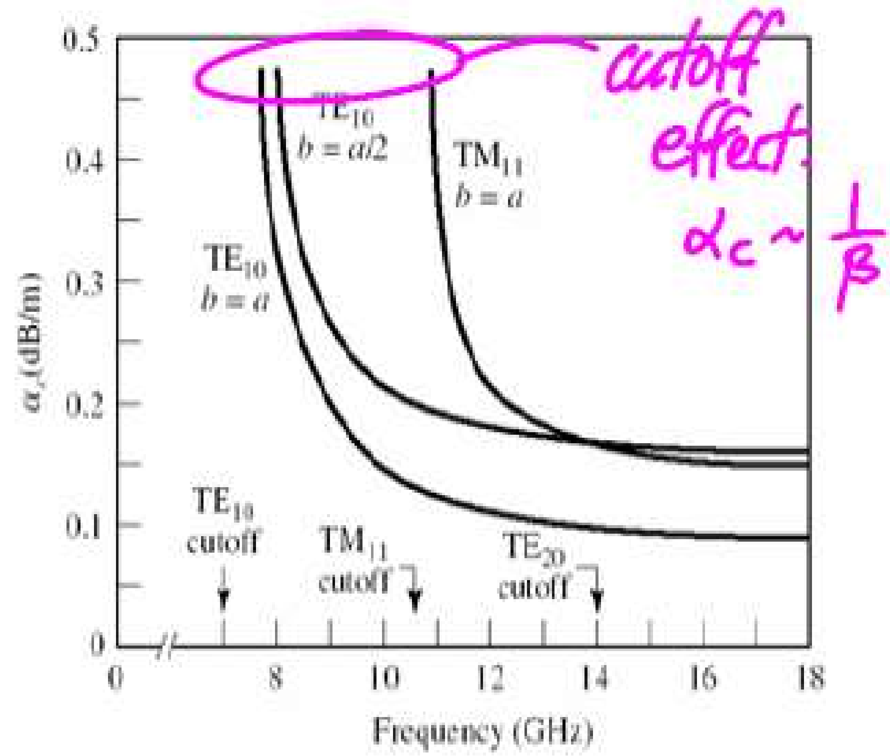


Figure 3.8 (p. 112)
Attenuation of various modes in a rectangular brass waveguide with a = 2.0 cm.

Attenuation for TE_{10} Mode (cont.)

Microwave Frequency Bands	
Letter Designation	**Frequency range**
L band	1 to 2 GHz
S band	2 to 4 GHz
C band	4 to 8 GHz
X band	8 to 12 GHz
K_u band	12 to 18 GHz
K band	18 to 26.5 GHz
K_a band	26.5 to 40 GHz
Q band	33 to 50 GHz
U band	40 to 60 GHz
V band	50 to 75 GHz
E band	60 to 90 GHz
W band	75 to 110 GHz
F band	90 to 140 GHz
D band	110 to 170 GHz

(from Wikipedia)

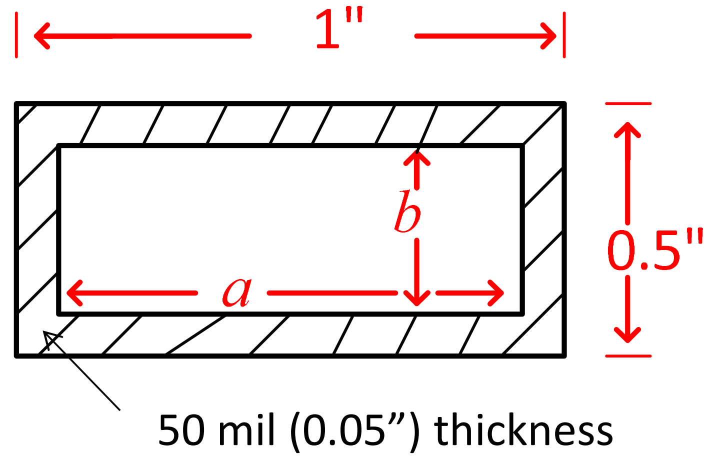
Modes in an X-Band Waveguide

$$a = 2.29\,\text{cm}\ (0.90")$$
$$b = 1.02\,\text{cm}\ (0.40")$$

Standard X-band waveguide (WR90)

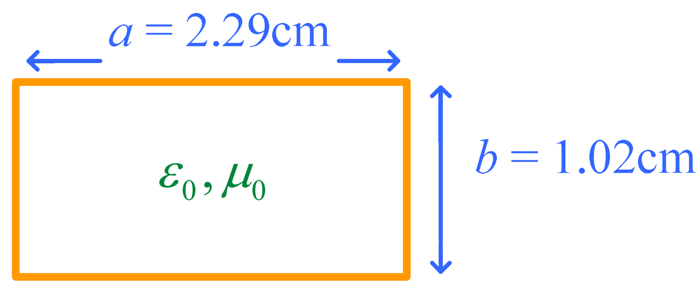
Mode	f_c [GHz]
TE_{10}	6.55
TE_{20}	13.10
TE_{01}	14.71
TE_{11}	16.10
TM_{11}	16.10
TE_{30}	19.65
TE_{21}	19.69
TM_{21}	19.69

X band: $\approx 8-12$ [GHz]



Example: X-Band Waveguide

Determine β and λ_g at 10 GHz and 6 GHz for the TE_{10} mode in an air-filled waveguide.



@ 10 GHz

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2} = \sqrt{\left(\frac{2\pi 10^{10}}{3\times 10^8}\right)^2 - \left(\frac{\pi}{0.0229}\right)^2}$$

$$\beta = 158.25\ [\text{rad/m}]$$

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{158.25} = 0.0397$$

$$\lambda_g = 3.97\ [\text{cm}]$$

Example: X-Band Waveguide (cont.)

@ 6 GHz

$$k_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2} = \sqrt{\left(\frac{2\pi 6\times 10^9}{3\times 10^8}\right)^2 - \left(\frac{\pi}{0.0229}\right)^2}$$

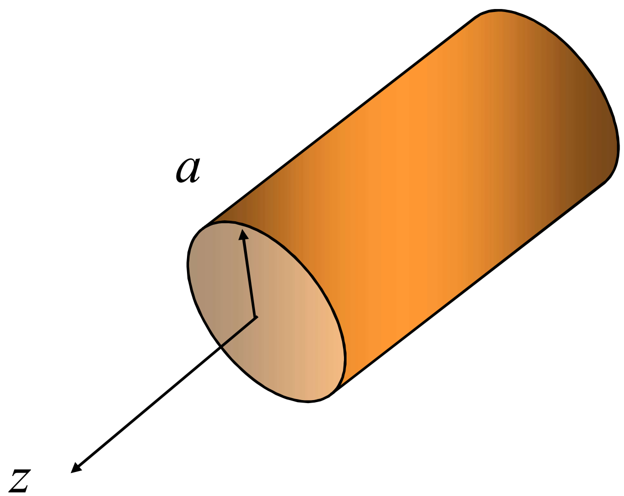
$$= -j\, 55.04 \;\; [1/\mathrm{m}]$$

$$\alpha = 55.04 \;\; [\mathrm{np/m}]$$
$$= 478.08 \; [\mathrm{dB/m}]$$

$$\lambda_g = \frac{2\pi}{\beta}$$

Evanescent mode: $\beta = 0$; λ_g is not defined!

Circular Waveguide



TM$_z$ mode:

$$\nabla^2 E_{z0}(\rho,\phi) + k_c^2 E_{z0}(\rho,\phi) = 0$$

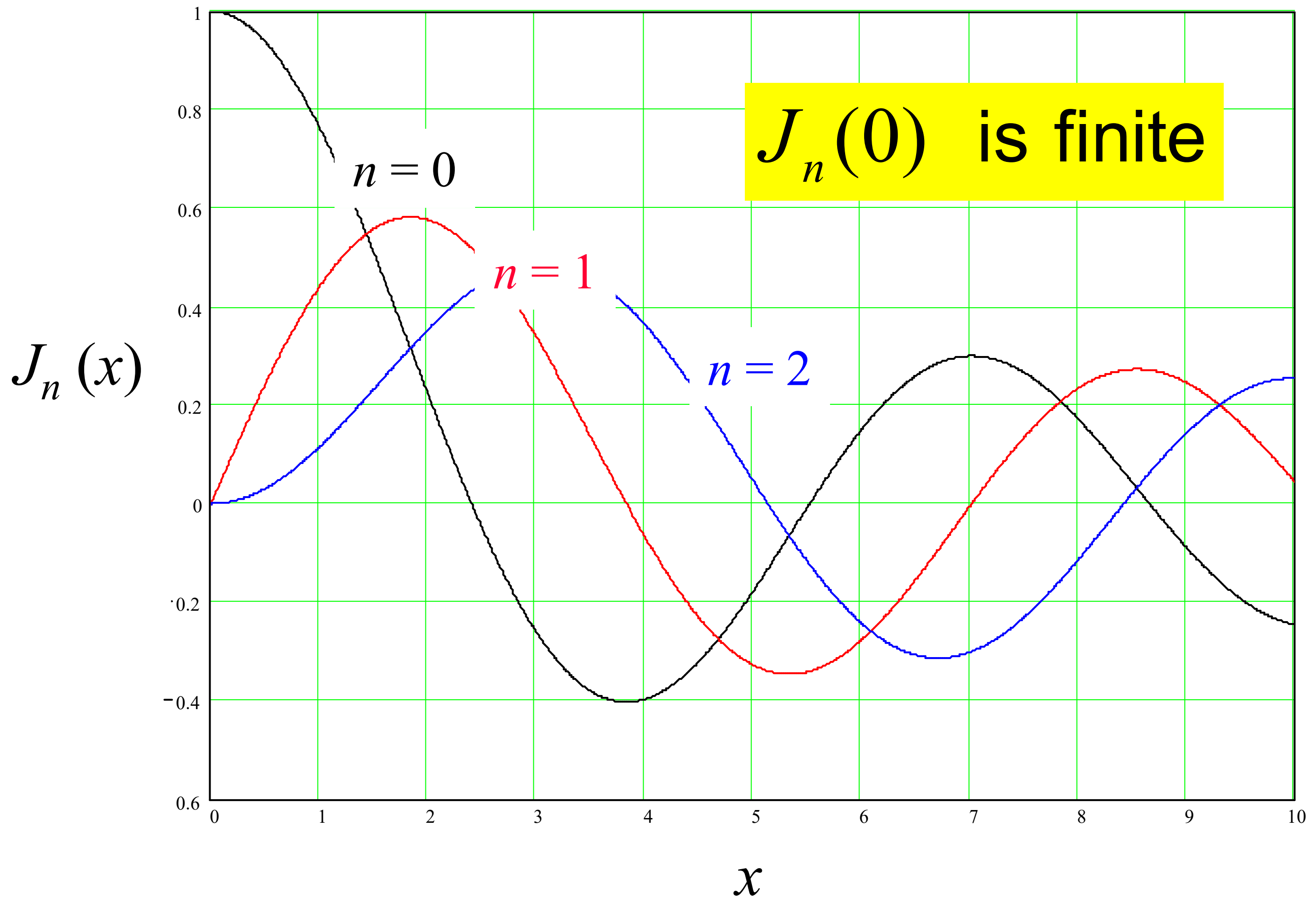
$$k_z^2 = k^2 - k_c^2$$

The solution in cylindrical coordinates is:

$$E_{z0}(\rho,\phi) = \begin{Bmatrix} J_n(k_c\rho) \\ Y_n(k_c\rho) \end{Bmatrix} \begin{Bmatrix} \sin(n\phi) \\ \cos(n\phi) \end{Bmatrix}$$

Note: The value n must be an integer to have unique fields.

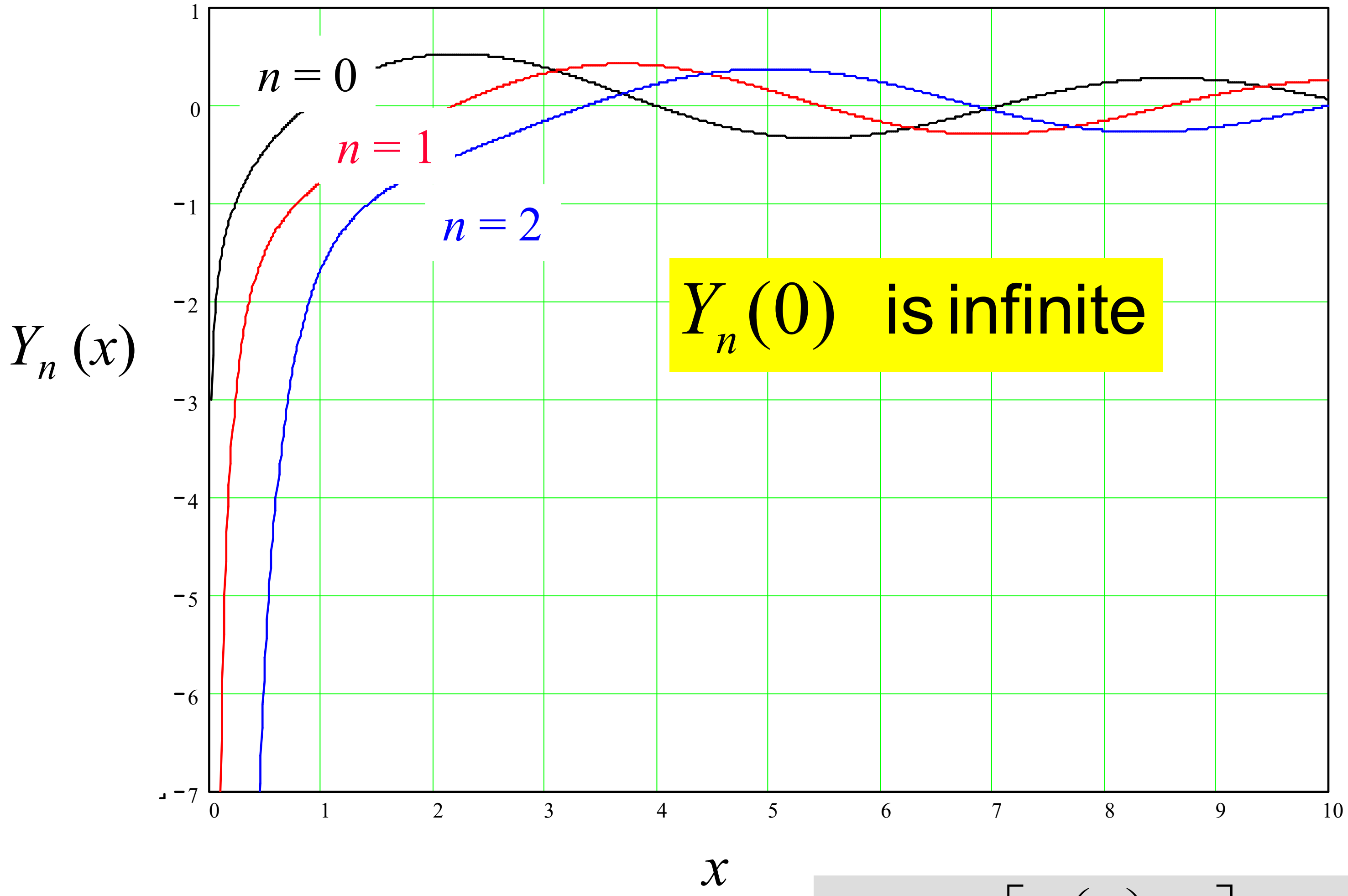
Plot of Bessel Functions



$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad x \to \infty$$

$$J_n(x) \sim x^n \left(\frac{1}{2^n n!}\right) \quad n = 0, 1, 2, \ldots., \quad x \to 0$$

Plot of Bessel Functions (cont.)



$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad x \to \infty$$

$$Y_0(x) \sim \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right], \ \gamma = 0.5772156, \ x \to 0$$

$$Y_n(x) \sim -\frac{1}{\pi}(n-1)!\left(\frac{2}{x}\right)^n, \ n = 1, 2, 3, \ldots\ldots, \ x \to 0$$

Circular Waveguide (cont.)

Choose (somewhat arbitrarily) $\cos(n\phi)$

$$E_z(\rho,\phi,z) = \begin{Bmatrix} J_n(k_c\rho) \\ Y_n(k_c\rho) \end{Bmatrix} \cos(n\phi)\; e^{\mp jk_z z}$$

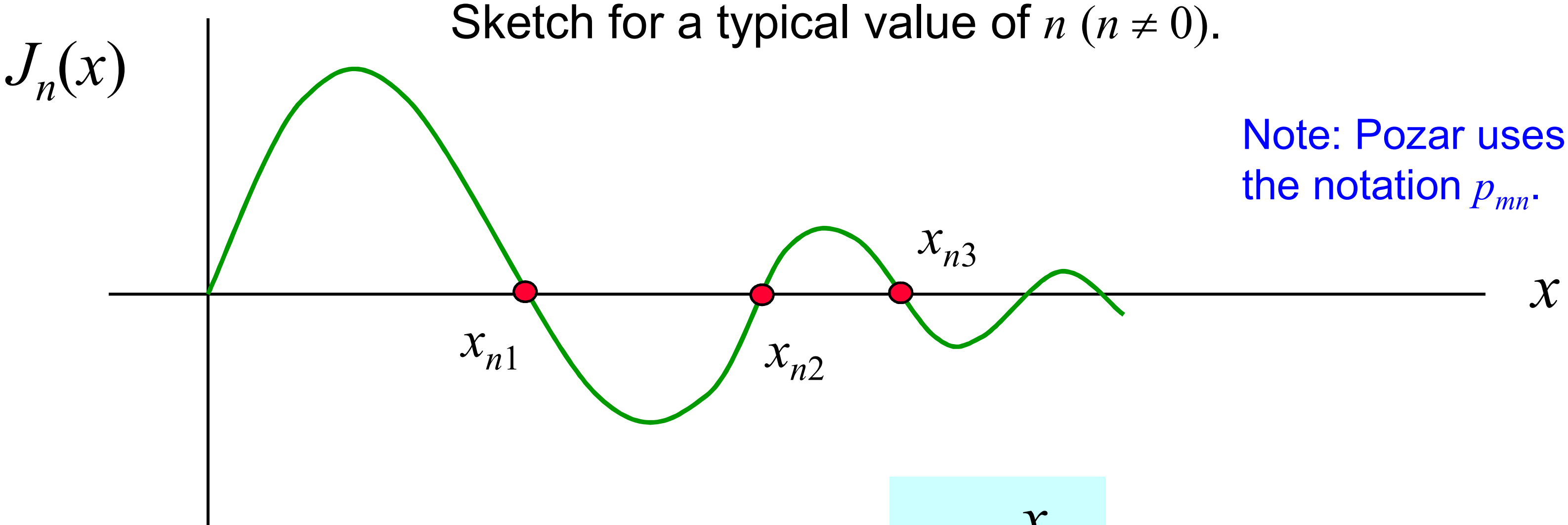
The field should be finite on the z axis

$\Rightarrow$ $Y_n(k_c\rho)$ is not allowed

$$E_z(\rho,\phi,z) = \cos(n\phi)\, J_n(k_c\rho)\, e^{\mp jk_z z}$$

Circular Waveguide (cont.)

B.C.'s: $E_z(a,\phi,z)=0$ Hence $J_n(k_c a)=0$



Sketch for a typical value of n $(n \neq 0)$.

Note: Pozar uses the notation p_{mn}.

$$k_c a = x_{np} \Rightarrow \boxed{k_c = \frac{x_{np}}{a}}$$

Note: The value $x_{n0}=0$ is not included since this would yield a trivial solution:

$$J_n\left(x_{n0}\frac{\rho}{a}\right)=J_n(0)=0$$

This is true unless $n=0$, in which case we cannot have $p=0$.

Circular Waveguide (cont.)

TM$_{np}$ mode:

$$E_z\left(\rho,\phi,z\right)=\cos(n\phi)\,J_n\left(x_{np}\frac{\rho}{a}\right)e^{\mp jk_z z}\qquad n=0,1,2\ldots$$

$$k_z=\sqrt{k^2-\left(\frac{x_{np}}{a}\right)^2}\qquad p=1,2,3,\ldots\ldots\ldots$$

Cutoff Frequency: TM$_z$

$$k_z^2 = k^2 - k_c^2$$

At $f = f_c$:

$$k_z = 0 \quad \Longrightarrow \quad k = k_c = \frac{x_{np}}{a}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{x_{np}}{a}$$

$$\boxed{f_c^{TM} = \left(\frac{c_d}{2\pi a}\right) x_{np}}$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

Cutoff Frequency: TM_z (cont.)

x_{np} values

$p \setminus n$	0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

TM_{01}, TM_{11}, TM_{21}, TM_{02}, ……..

TE$_z$ Modes

Proceeding as before, we now have that

$$H_z\left(\rho,\phi,z\right)=\cos(n\phi)\,J_n(k_c\rho)\,e^{\mp jk_z z}$$

Set $\quad E_\phi\left(a,\phi,z\right)=0$

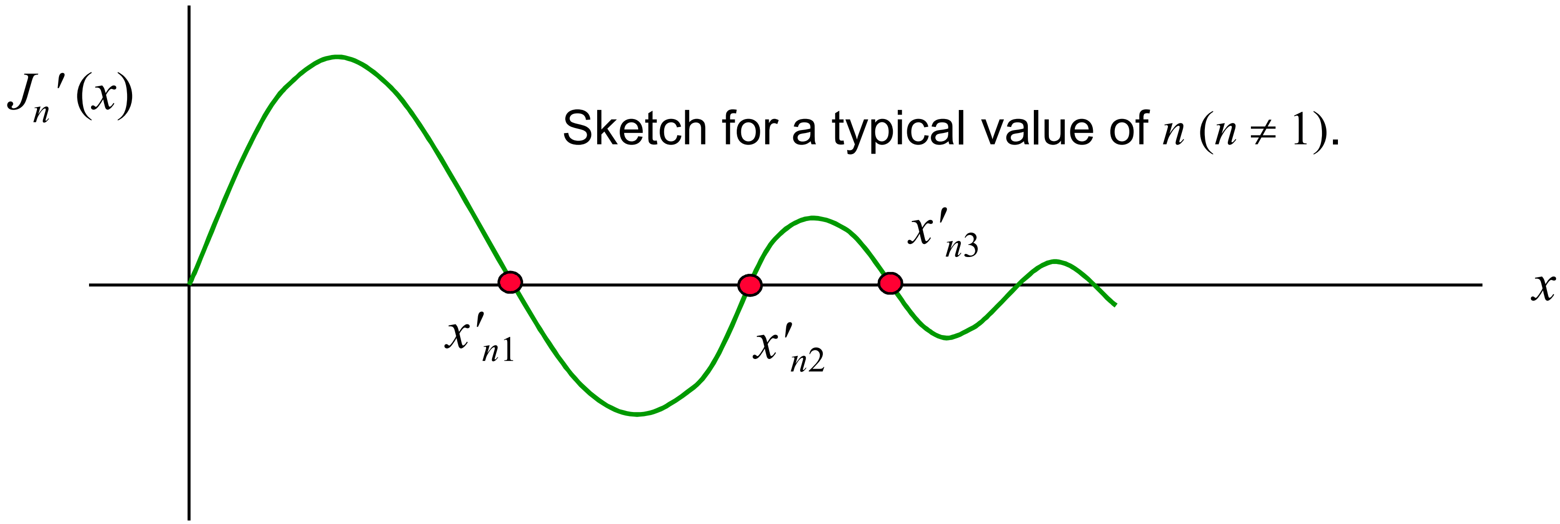
$$E_\phi=\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho}$$ (From Ampere's law)

$$\Rightarrow \frac{\partial H_z}{\partial\rho}=0\Big|_{\rho=a}$$

Hence $\quad J_n'(k_c a)=0$

TE$_z$ Modes (cont.)

$$J_n'(k_c a) = 0$$



Sketch for a typical value of n ($n \neq 1$).

$$k_c a = x'_{np}$$

$$k_c = \frac{x'_{np}}{a} \qquad p = 1, 2, 3, ...$$

We don't need to consider $p = 0$; this is explained on the next slide.

TE$_z$ Modes (cont.)

$$H_z(\rho,\phi,z) = \cos(n\phi)\, J_n\left(x'_{np}\frac{\rho}{a}\right) e^{\mp jk_z z} \qquad p = 1, 2, \ldots$$

Note: If $p = 0$ $\quad x'_{np} = 0$

We then have, for $p = 0$:

$$n \neq 0 \quad J_n\left(x'_{np}\frac{\rho}{a}\right) = J_n(0) = 0 \qquad \text{(trivial solution)}$$

$$n = 0 \quad J_0\left(x'_{np}\frac{\rho}{a}\right) = J_0(0) = 1$$

$$\Rightarrow \quad \underline{H} = \underline{\hat{z}}\, e^{\mp jk_z z} = \underline{\hat{z}}\, e^{\mp jkz} \qquad \text{(nonphysical solution)}$$

(The TE$_{00}$ mode is not physical.)

Cutoff Frequency: TE$_z$

$$k_z^2 = k^2 - k_c^2$$

$$k_z = 0 \quad \Rightarrow \quad k_c = k = \frac{x'_{np}}{a}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{x'_{np}}{a}$$

Hence

$$\boxed{f_c^{TE} = \left(\frac{c_d}{2\pi a}\right) x'_{np}}$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

Cutoff Frequency: TE_z

x'_{np} values

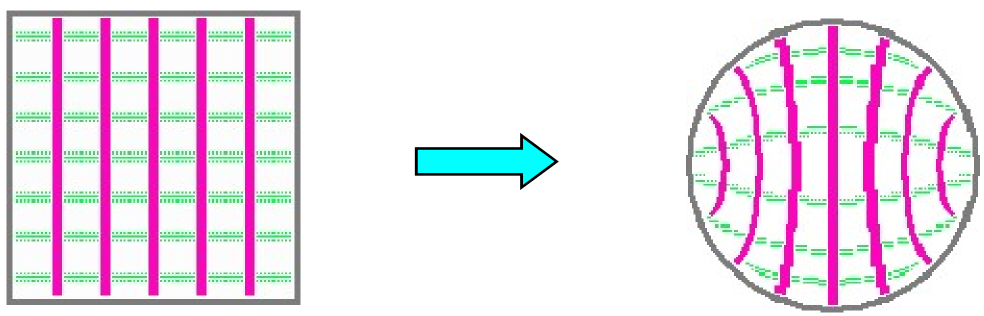
$p \setminus n$	0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	5.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

TE_{11}, TE_{21}, TE_{01}, TE_{31}, ……..

TE_{11} Mode

The dominant mode of circular waveguide is the TE_{11} mode.

Electric field
Magnetic field



(From Wikipedia)

TE_{10} mode of rectangular waveguide

TE_{11} mode of circular waveguide

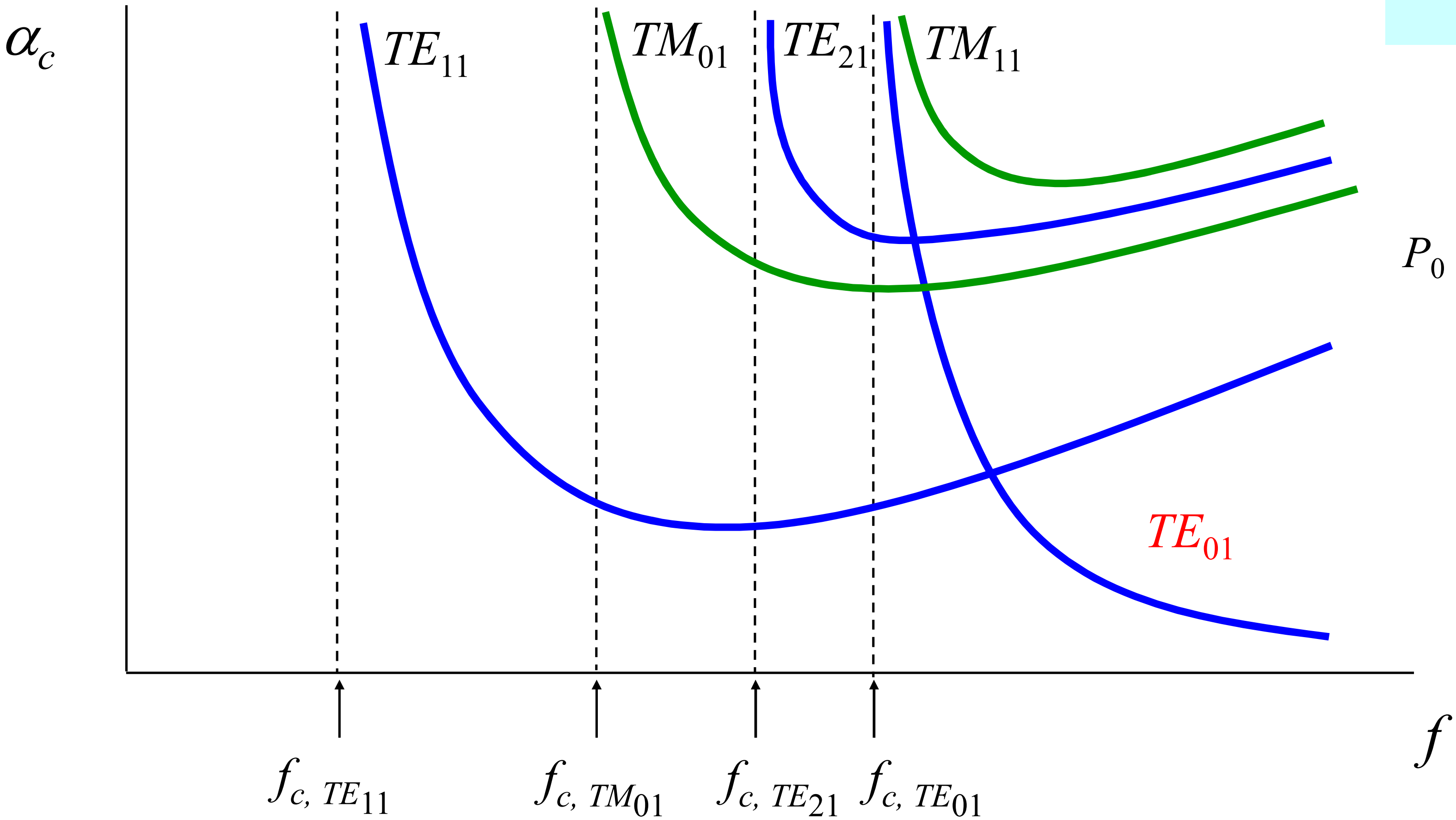
The mode can be thought of as an evolution of the TE_{10} mode of rectangular waveguide as the boundary changes shape.

TE_{01} Mode

The TE_{01} mode has the unusual property that the conductor attenuation decreases with frequency. (With most waveguide modes, the conductor attenuation increases with frequency.)

The TE_{01} mode was studied extensively as a candidate for long-range communications – but eventually fiber-optic cables became available with even lower loss. It is still useful for some high-power applications.

TE_{01} Mode (cont.)



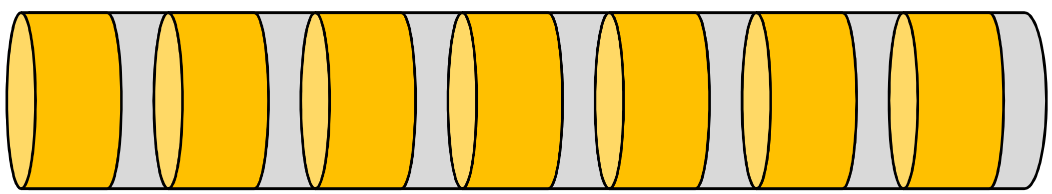
$$\alpha_c = \frac{P_l(0)}{2\boxed{P_0}}$$

$P_0 = 0$ at cutoff

TE_{01} Mode (cont.)

Practical Note:

The TE_{01} mode has only an azimuthal (ϕ-directed) surface current on the wall of the waveguide. Therefore, it can be supported by a set of conducting rings, while the lower modes (TE_{11} ,TM_{01}, TE_{21}, TM_{11}) will not propagate on such a structure.



(A helical spring will also work fine.)

TE_{01} Mode (cont.)

From the beginning, the most obvious application of waveguides had been as a communications medium. It had been determined by both Schelkunoff and Mead, independently, in July 1933, that an axially symmetric electric wave (TE_{01}) in circular waveguide would have an attenuation factor that decreased with increasing frequency [44]. This unique characteristic was believed to offer a great potential for wide-band, multichannel systems, and for many years to come the development of such a system was a major focus of work within the waveguide group at BTL. It is important to note, however, that the use of waveguide as a long transmission line never did prove to be practical, and Southworth eventually began to realize that the role of waveguide would be somewhat different than originally expected. In a memorandum dated October 23, 1939, he concluded that microwave radio with highly directive antennas was to be preferred to long transmission lines. "Thus," he wrote, “we come to the conclusion that the hollow, cylindrical conductor is to be valued primarily as a new circuit element, but not yet as a new type of toll cable” [45]. It was as a circuit element in military radar that waveguide technology was to find its first major application and to receive an enormous stimulus to both practical and theoretical advance.

K. S. Packard, “The Origins of Waveguide: A Case of Multiple Rediscovery,” IEEE Trans. MTT, pp. 961-969, Sept. 1984.